

PERMUTATION REPRESENTATIONS ON SCHUBERT VARIETIES

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ABSTRACT. This paper defines and studies permutation representations on the equivariant cohomology of Schubert varieties, as representations both over \mathbb{C} and over $\mathbb{C}[t_1, t_2, \dots, t_n]$. We show these group actions are the same as an action of simple transpositions studied geometrically by M. Brion, and give topological meaning to the divided difference operators of Bernstein-Gelfand-Gelfand, Demazure, Kostant-Kumar, and others. We analyze these representations using the combinatorial approach to equivariant cohomology introduced by Goresky-Kottwitz-MacPherson. We find that each permutation representation on equivariant cohomology produces a representation on ordinary cohomology that is trivial, though the equivariant representation is not.

1. INTRODUCTION

Geometric representation theory is an important approach to understanding permutation representations. It builds algebraic varieties whose cohomology carries group actions and uses the geometric structure to analyze the representations. However, constructions of these representations to date are either not elementary (e.g. [Sp], [BoM], or [L]) or not explicit (e.g. [S] or [P]). We rectify this situation. This paper constructs permutation representations on cohomology and equivariant cohomology of Schubert varieties in a simple yet concrete fashion. Divided difference operators akin to those of Bernstein-Gelfand-Gelfand [BGG] and Demazure [D] result naturally from this geometric representation.

Denote the flag variety by the quotient $GL_n(\mathbb{C})/B$, where B is the group of invertible upper-triangular matrices. Let $[g]$ be the flag corresponding to the matrix g . For each permutation matrix w , the Schubert variety X_w is the closure of the flags $[Bw]$ in $GL_n(\mathbb{C})/B$. Schubert varieties are studied because they form a natural basis for the (equivariant) cohomology of the flag variety, and have deep combinatorial connections. The diagonal matrices in $GL_n(\mathbb{C})$ act on each Schubert variety: if t is diagonal and $[g]$ is in X_w then $t \cdot [g] = [tg]$. We study $H_T^*(X_w)$ for this torus.

This paper considers the action of the permutation group S_n on equivariant and ordinary cohomology induced from the geometric action of $u \in S_n$ on $[g] \in GL_n(\mathbb{C})/B$ given by $u \cdot [g] = [u^{-1}g]$. Corollary 2.10 provides a simple formula for the action of an *arbitrary* element of S_n on each equivariant class in $H_T^*(GL_n(\mathbb{C})/B)$. We then obtain an explicit formula for the action of a simple transposition on the basis of equivariant Schubert classes, which is the core of our work. If $s_{i,i+1} \in S_n$

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is the simple transposition $(i, i+1)$ and $[\Omega_v]_{X_w}$ is a Schubert class in X_w then

$$(1) \quad s_{i,i+1}[\Omega_v]_{X_w} = \begin{cases} [\Omega_v]_{X_w} & \text{if } s_{i,i+1}v > v \text{ and} \\ [\Omega_v]_{X_w} + (t_{i+1} - t_i)[\Omega_{s_{i,i+1}v}]_{X_w} & \text{if } s_{i,i+1}v < v. \end{cases}$$

This formula was proven by M. Brion in [B, Proposition 6.2] using geometric methods that can only construct the action of simple transpositions on $H_T^*(X_w)$. We show the action in [B] derives from the global geometric action of S_n on $GL_n(\mathbb{C})/B$.

Equivariant cohomology surjects onto ordinary cohomology by the map that sends each $t_i \mapsto 0$. Together with Equation (1), this is part of the main theorem:

Theorem 1.1. *Fix a permutation w . Denote the trivial representation in degree d by 1^d . The action of $u \in S_n$ on $[g] \in GL_n(\mathbb{C})/B$ given by $u \cdot [g] = [u^{-1}g]$ induces an action of S_n on both $H_T^*(X_w)$ and $H^*(X_w)$ with the following properties:*

- (1) $H_T^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [S_n] \cap X_w} 1^{\ell(v)} \otimes \mathbb{C}[t_1, \dots, t_n]$ as a graded $\mathbb{C}[S_n]$ -module, where S_n acts on $\mathbb{C}[t_1, \dots, t_n]$ by permuting the variables;
- (2) $H_T^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [S_n] \cap X_w} 1^{\ell(v)}$ as a graded twisted module over $\mathbb{C}[t_1, \dots, t_n][S_n]$; and
- (3) $H^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [S_n] \cap X_w} 1^{\ell(v)}$ as a graded $\mathbb{C}[S_n]$ -module.

Since the group $GL_n(\mathbb{C})$ is connected, the endomorphism on $GL_n(\mathbb{C})/B$ given by $u \cdot [g] = [u^{-1}g]$ is homotopic to the identity for each $u \in GL_n(\mathbb{C})$, including $u \in S_n$. Thus the action induced by S_n on the ordinary cohomology $H^*(GL_n(\mathbb{C})/B)$ is trivial. However, the map $u \cdot [g] = [u^{-1}g]$ is not T -equivariant, so this fails for equivariant cohomology (indeed, Equation (1) shows S_n does not act trivially on equivariant Schubert classes). Moreover, the action $u \cdot [g] = [u^{-1}g]$ is not generally well-defined on the Schubert variety X_w . In fact S_n does act on the cohomology of Schubert varieties and this action is trivial because it is a quotient of a direct sum of trivial representations. An open question is to interpret this action geometrically.

Rewriting Equation (1) gives the divided difference operator $[X] \mapsto \frac{[X] - s_{i,i+1}[X]}{t_i - t_{i+1}}$. First defined by Bernstein-Gelfand-Gelfand [BGG] and Demazure [D], divided difference operators have been widely used to analyze the algebraic structure of $H^*(GL_n(\mathbb{C})/B)$ (e.g. [KK1], [KK2]) and to find Schubert polynomials, namely “nice” representatives for Schubert classes (e.g. [LS], [BJS], [FK]); [F2] has a survey. This paper is atypical in treating $H_T^*(GL_n(\mathbb{C})/B)$ as a *subring* of a product of $n!$ polynomial rings rather than as a quotient ring (see also [KK1], [KK2], and [B]). The group S_n acts naturally on $H_T^*(GL_n(\mathbb{C})/B)$ by both left and right multiplication on the index set, though this distinction is often elided in the literature. The two actions give divided difference operators in very different ways. We consider the action of left multiplication and its left divided difference operator; the other case is in [T2]. The formula for the left divided difference operator is that of Bernstein-Gelfand-Gelfand/Demazure, but the morphism is not. This case is also studied in [B], where it is misidentified as the Bernstein-Gelfand-Gelfand/Demazure operator. In fact, the right divided difference operators—defined by Kostant-Kumar in [KK1], [KK2]—were proven in [A] to be the divided difference operators of Bernstein-Gelfand-Gelfand and Demazure. If $H_T^*(GL_n(\mathbb{C})/B)$ is presented as a quotient ring, the left divided difference operator is the divided difference operator “in the y -variables” of double Schubert polynomials (see e.g. [F1]). Unusually, our arguments rely primarily on elementary combinatorics.

Our approach to equivariant cohomology uses the method of M. Goresky, R. Kot-twitz, and R. MacPherson [GKM], which translates topological data of cohomology into a purely combinatorial calculation and is sketched in Section 2.2. ([KK1] and [KK2] essentially develop GKM theory by hand for $GL_n(\mathbb{C})/B$.) The GKM method applies to Schubert varieties, as described in [C]. The ring $H_T^*(X_w)$ has a special basis of “Schubert classes” produced by a combinatorial algorithm modeled on work of Guillemin-Zara and Knutson-Tao [GZ1], [GZ2], [KT]. These classes are also the Kostant-Kumar ξ^v -classes of [KK1] and [KK2]. Section 2.3 discusses these bases for more general algebraic varieties than in [GZ1], [GZ2]. Section 3 contains the equivariant cohomology calculations.

Our statements and proofs are first given for flags over $GL_n(\mathbb{C})$. Nonetheless, these results hold for all Lie types. Our presentation was chosen because the exposition is more concrete for $GL_n(\mathbb{C})$, because some readers will primarily be interested in this case, and because the reader interested in other Lie types can usually extrapolate those results immediately from our description of the special case. The general statements and all proofs that do not just change notation are in Section 4.

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2. BACKGROUND

2.1. Permutation statistics. We fix notation and give background about flag varieties and permutations.

As before, B denotes the group of invertible upper-triangular matrices. The flag variety is denoted G/B , and its typical element is denoted $[g]$. We also consider $[g]$ to be the collection of nested subspaces whose i -dimensional subspace is the span of the first i columns of g , for each i .

Fix the standard basis e_1, e_2, \dots, e_n in \mathbb{C}^n . Each permutation matrix w also gives an element $[w]$ of the flag variety. We use the same notation for the matrix w and for the permutation on $\{1, 2, \dots, n\}$ given by $we_i = e_{w(i)}$. We write s_{jk} to denote the transposition that exchanges j and k .

If u is a matrix, its (i, j) entry is denoted u_{ij} .

Definition 2.1. For each permutation w , define the subgroup U_w of GL_n to be

$$\{u \in B : u_{ii} = 1 \text{ for each } i, \text{ and if } i \neq j \text{ then } u_{ij} = 0 \text{ unless } w^{-1}(i) > w^{-1}(j)\}.$$

The nonzero, nondiagonal entries in U_w are important in what follows.

Definition 2.2. A pair (i, j) is an inversion for w if $i < j$ and $w^{-1}(i) > w^{-1}(j)$. The set $\mathcal{I}_w = \{t_i - t_j : i < j, w^{-1}(i) > w^{-1}(j)\}$ is a collection of binomials bijectively associated to the inversions for w .

More generally, \mathcal{I}_w is the set of the positive roots that w sends to negative roots.

Definition 2.3. For each $j \neq k$ and $c \in \mathbb{C}$, define the matrix $G_{jk}(c)$ by

$$(G_{jk}(c))_{il} = \begin{cases} 1 & \text{if } i = l, \\ c & \text{if } i = j \text{ and } k = l, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The group $\{G_{jk}(c) : c \in \mathbb{C}\}$ is a subgroup of U_w if $j < k$ and $w^{-1}(j) > w^{-1}(k)$. Lie theoretically, the group $\{G_{jk}(c) : c \in \mathbb{C}\}$ is a root subgroup.

If w is a permutation, then $[Bw]$ is the Schubert cell corresponding to w and $X_w = \overline{[Bw]}$ is the corresponding Schubert variety. For each w , the Schubert cell $[Bw]$ is homeomorphic to affine space and is parametrized by the subgroup U_w . The flag variety is the disjoint union $\bigcup_{w \in S_n} [Bw]$.

Proposition 2.4. *Each flag in the Schubert cell $[Bw]$ can be written uniquely as a matrix of the form $U_w w$. Each matrix $g \in U_w w$ can be written uniquely as $g = w + u$, where u is zero except in entries u_{ij} that are **both** to the left **and** above a nonzero entry in w .*

If w and v are permutations, we say that $w \geq v$ if $\overline{[Bw]} \supseteq [Bv]$. The length of the permutation w is $\ell(w) = \dim \overline{[Bw]}$. We have $\ell(w) = |\mathcal{I}_w|$ by construction.

The combinatorial argument in the next proof is similar to [KM, Section 3]. Similar results appear in the literature (especially [BGG, Lemma 2.4]); since we were unable to find this formulation, we include it here. Section 4 contains the general proof.

Lemma 2.5. *If $j < k$ and w is a permutation with $\ell(s_{jk}w) = \ell(w) + 1$ then*

- (1) *the inversions $\mathcal{I}_{s_{jk}w} \cong \{t_j - t_k\} \cup \mathcal{I}_w \bmod(t_j - t_k)$ with multiplicity; and*
- (2) *if $s_{i,i+1}$ is a simple transposition with $s_{i,i+1}w > w$ then $s_{i,i+1}s_{jk}w > s_{jk}w$.*

Proof. We use Proposition 2.4 repeatedly. Figure 1 is a schematic for the matrices

$$\begin{pmatrix} & A & & B & \\ \hline C & 1 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ & 0 & & & \\ & \vdots & & D & \\ & 0 & & & \\ \hline E & 0 & F & 1 & 0 \cdots 0 \\ & 0 & & 0 & \\ & \vdots & & \vdots & \\ & 0 & & 0 & \end{pmatrix} \quad \begin{pmatrix} & A & & B & \\ \hline C & \mathbf{a} & F' & 1 & 0 \cdots 0 \\ & & & 0 & \\ & D' & & \vdots & \\ & & & 0 & \\ \hline E & 1 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ & 0 & & 0 & \\ & \vdots & & \vdots & \\ & 0 & & 0 & \end{pmatrix}$$

$U_w w$ $U_{s_{jk}w} s_{jk} w$

FIGURE 1. Transposing $U_w w$

$U_{s_{jk}w} s_{jk} w$ and $U_w w$. We first confirm that the matrices are labeled correctly. The matrices differ only in the columns and rows indicated in Figure 1. Regions A , B , C , and E have the same free entries in $U_w w$ as in $U_{s_{jk}w} s_{jk} w$. The entry labeled \mathbf{a} is free in one set but not in the other. Region D' has all of the nonzero entries from D , plus perhaps additional nonzero entries; similarly for F' and F . We conclude that the matrices on the right are translated by a permutation of greater length than those on the left, which means they must be $U_{s_{jk}w} s_{jk} w$.

Now compare U_w to $U_{s_{jk}w}$ in Figure 2. If $\ell(s_{jk}w) = \ell(w) + 1$ then in fact D and D' have the same nonzero entries, as do F and F' . The entry marked \mathbf{a} is free in $U_{s_{jk}w}$ but not in U_w , so $\mathcal{I}_{s_{jk}w}$ contains $t_j - t_k$ while \mathcal{I}_w does not. Other than $t_j - t_k$, the multisets $\mathcal{I}_{s_{jk}w} \bmod(t_j - t_k)$ and $\mathcal{I}_w \bmod(t_j - t_k)$ are the same.

$$\begin{array}{c}
\left(\begin{array}{c|c|c|c|c} & A & & B & \\ \hline 0 \cdots 0 & 1 & C & 0 & C \\ \hline & 0 & & & \\ & \vdots & & D & \\ & 0 & & & \\ \hline 0 \cdots 0 & 0 & 0 \cdots 0 & 1 & E, F \\ \hline & 0 & & 0 & \\ & \vdots & & \vdots & \\ & 0 & & 0 & \end{array} \right) \\
U_w
\end{array}
\qquad
\begin{array}{c}
\left(\begin{array}{c|c|c|c|c} & B & & A & \\ \hline 0 \cdots 0 & 1 & C, F' & \mathbf{a} & C, F' \\ \hline & 0 & & & \\ & \vdots & & D' & \\ & 0 & & & \\ \hline 0 \cdots 0 & 0 & 0 \cdots 0 & 1 & E \\ \hline & 0 & & 0 & \\ & \vdots & & \vdots & \\ & 0 & & 0 & \end{array} \right) \\
U_{s_{jk}w}
\end{array}$$

FIGURE 2. The subgroups U_w and $U_{s_{jk}w}$

For the last part, we also use Figure 2. We are given that entry $(i, i+1)$ is zero in U_w and need to show that entry $(i, i+1)$ is zero in $U_{s_{jk}w}$. If $(i, i+1)$ is in region A in U_w then entry (i, k) in region B of U_w is also zero, according to the description of the matrices $U_w w$ in Proposition 2.4. If $(i, i+1)$ is in region C or E in U_w then it is unchanged in $U_{s_{jk}w}$. This is true also if $(i, i+1)$ is in region D of U_w , since the free entries of D and D' are exactly the same. If $(i, i+1)$ is in region F , then entry $(i, i+1)$ is zero in $U_{s_{jk}w}$ by construction. If $(i, i+1)$ is in region F' then to be above the diagonal it must be between the j^{th} and k^{th} columns. In that case, it is zero in $U_{s_{jk}w}$ since F' and F have the same free columns. No other free entry in U_w differs from that in $U_{s_{jk}w}$. \square

The next lemma specializes the previous argument.

Lemma 2.6. *If $s_{i,i+1}w > w$, the inversions satisfy $\mathcal{I}_{s_{i,i+1}w} = \{t_i - t_{i+1}\} \cup s_{i,i+1}\mathcal{I}_w$, where $s_{i,i+1}$ acts on $\mathbb{C}[t_1, \dots, t_n]$ by $s_{i,i+1}(t_l) = t_{s_{i,i+1}(l)}$ for each l .*

Proof. Consider Lemma 2.5.1 in the case when $s_{jk} = s_{i,i+1}$. The free entries in regions C and E of Figure 1 are exactly the same, and the regions marked D and D' do not exist. Figure 2 now looks like:

$$\left(\begin{array}{c|c|c|c} A & B & & \\ \hline 0 \cdots 0 & 1 & 0 & C \\ \hline 0 \cdots 0 & 0 & 1 & C, F \\ \hline \vdots & \vdots & & \end{array} \right)
\qquad
\left(\begin{array}{c|c|c|c} B & A & & \\ \hline 0 \cdots 0 & 1 & \mathbf{a} & C, F \\ \hline 0 \cdots 0 & 0 & 1 & C \\ \hline \vdots & \vdots & & \end{array} \right)$$

The inversions in columns i and $i+1$ and rows i and $i+1$ are exchanged. \square

2.2. GKM theory. The GKM method reduces the task of identifying equivariant cohomology of a suitable space \mathcal{X} to an algebraic computation on a combinatorial graph associated to \mathcal{X} . We sketch this method; the reader interested in more is encouraged to see the original paper [GKM], the survey [T1], or the very nice presentation in the introduction of [KT].

Let \mathcal{X} be a complex projective algebraic variety with a linear algebraic action of a torus $T = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ that satisfies the following conditions: the torus has finitely many fixed points as well as finitely many one-dimensional orbits in \mathcal{X} ; and

\mathcal{X} is *equivariantly formal*, a technical property that holds if, for instance, \mathcal{X} has no odd-dimensional (ordinary) cohomology. These are the GKM conditions. Any \mathcal{X} satisfying the GKM conditions is called a GKM variety.

The group G acts on the flag variety G/B by $h \cdot [g] = [hg]$. If T is the torus consisting of all diagonal matrices in G , the G -action restricts to a T -action. Many subvarieties of the flag variety do not carry this torus action. However, Schubert varieties do and are equivariantly formal with respect to this T -action. We give the following result for completeness, though it is not new.

Proposition 2.7. *Every Schubert variety is T -equivariant and is equivariantly formal with respect to this action.*

Proof. Each Schubert variety has a cell decomposition as a union of Schubert cells. Each Schubert variety is T -closed since it is B -closed. No Schubert variety has odd-dimensional cohomology since each $[Bw]$ is a complex affine cell. Consequently, every Schubert variety is equivariantly formal with respect to every possible torus action by [GKM, Theorem 14]. \square

When \mathcal{X} satisfies the GKM conditions, the closure of each one-dimensional T -orbit in \mathcal{X} is homeomorphic to \mathbb{P}^1 . Both the origin and the point at infinity of \mathbb{P}^1 correspond to T -fixed points in \mathcal{X} . If O is a one-orbit in \mathcal{X} and N_O and S_O are the T -fixed points in \overline{O} , then the torus acts on the tangent space $\mathcal{T}_{N_O}(\overline{O})$ with weight α if and only if the torus acts on the tangent space $\mathcal{T}_{S_O}(\overline{O})$ with weight $-\alpha$ (see [GKM, 7.1.1]).

In this case, we associate to \mathcal{X} a labeled directed graph called the moment graph of \mathcal{X} . The vertices of the moment graph of \mathcal{X} are the T -fixed points in \mathcal{X} , denoted \mathcal{X}^T . If v and w are two vertices, there is an edge between v and w exactly when there is a one-orbit in \mathcal{X} whose closure contains both v and w . Given a directed edge $v \rightarrow w$ with associated one-orbit O , the edge $v \rightarrow w$ is labeled with the weight of the torus action on the tangent space $\mathcal{T}_v(\overline{O})$. (The moment graph of \mathcal{X} is not canonically directed; see Section 2.3 for more.)

The flag variety is the example that forms the basis of the calculations in this paper. Part 1 of the next proposition describes the moment graph for the flag variety GL_n/B , as stated (in more generality) in [C, Theorem F]. Figure 3 gives the moment graph when $n = 3$, drawn so that each edge is directed from the higher endpoint to the lower endpoint.



FIGURE 3. The moment graph for G/B when $n = 3$

Several corollaries follow immediately from the description of the moment graph. Recall that if V' is a subset of the vertices of the graph (V, E) , then the subgraph *induced* by V' is the maximal subgraph of (V, E) with vertex set V' , namely the graph (V', E') where $E' = \{vw : v, w \in V' \text{ and } vw \in E\}$.

Proposition 2.8. *In $GL_n(\mathbb{C})/B$:*

- (1) The vertices of the moment graph for GL_n/B are $[w]$ for $w \in S_n$. If $j < k$ then the flags $[G_{jk}(c)w]$ with $c \neq 0$ correspond to an edge directed out of w if and only if $w^{-1}(j) > w^{-1}(k)$. This edge is directed into $s_{jk}w$ and is labeled $t_j - t_k$.
- (2) There are $\ell(w)$ edges directed out of w , labeled exactly by the set \mathcal{I}_w .
- (3) If there is an edge from $s_{i,i+1}w$ to w and a transposition $s_{jk} \neq s_{i,i+1}$ with both $\ell(s_{jk}w) = \ell(w) + 1$ and an edge from $s_{jk}w$ to w , then there is also an edge from $s_{i,i+1}s_{jk}w$ to $s_{jk}w$.
- (4) If $s_{i,i+1}w > w$ then the edges directed out of $s_{i,i+1}w$ are labeled bijectively with $\{t_i - t_{i+1}\} \cup s_{i,i+1}\mathcal{I}_w$, where $s_{i,i+1}$ acts on $\mathbb{C}[t_1, \dots, t_n]$ by the rule $s_{i,i+1}(t_l) = t_{s_{i,i+1}(l)}$ for each l .
- (5) The moment graph for the Schubert variety X_w is the subgraph of the moment graph for GL_n/B induced by the permutation flags in X_w , namely $[v]$ where $v \in S_n$ satisfies $v \leq w$.

Proof. Part 1 is [C, Theorem F], with our convention for directing and labeling the graph. The number of edges directed out of w is exactly $\dim \overline{Bw} = \ell(w)$, and is indexed by the inversions for w . This proves Part 2. Part 3 is Lemma 2.5.2. Part 4 is Lemma 2.6. Part 5 is in [C, Theorem F5] and is a nice exercise for the reader. \square

The main theorem that we use is:

Proposition 2.9. (Goresky, Kottwitz, MacPherson) *Let \mathcal{X} be a GKM variety and let S be the polynomial ring $\mathbb{C}[t_1, \dots, t_n]$. For each one-dimensional orbit O , the fixed points in \overline{O} are denoted N_O and S_O , and the weight of the T -action on O is t_O . The equivariant cohomology $H_T^*(\mathcal{X})$ is the subring of $S^{|\mathcal{X}^T|}$ given by:*

$$H_T^*(\mathcal{X}) = \left\{ (p_w)_{w \in \mathcal{X}^T} \in S^{|\mathcal{X}^T|} : \text{for each one-orbit } O, p_{N_O} - p_{S_O} \in \langle t_O \rangle \right\}.$$

An element $u \in S_n$ acts on the flag $[g] \in G/B$ by $u \cdot [g] = [u^{-1}g]$. This geometric action induces an action of S_n on the equivariant cohomology of G/B as follows. Recall that $u \in S_n$ acts on the polynomial ring $\mathbb{C}[t_1, \dots, t_n]$ by $u \cdot p(t_1, \dots, t_n) = p(t_{u(1)}, \dots, t_{u(n)})$.

Corollary 2.10. *The action of S_n on G/B induces a well-defined group action of S_n on the equivariant cohomology $H_T^*(G/B)$, given by the rule that if $u \in S_n$ and $p = (p_v)_{v \in S_n} \in H_T^*(G/B)$ then the localization of $u \cdot p$ at each $v \in S_n$ is*

$$(u \cdot p)_v(t_1, \dots, t_n) = u \cdot p_{u^{-1}v}(t_1, \dots, t_n) = p_{u^{-1}v}(t_{u(1)}, \dots, t_{u(n)}).$$

Proof. The action of $u \in S_n$ on the variety G/B gives a graph automorphism of the undirected moment graph since $s_{u^{-1}(j), u^{-1}(k)} u^{-1}v = u^{-1}s_{jk}v$. In particular, there is an edge between v and $s_{jk}v$ if and only if there is an edge between $u^{-1}v$ and $u^{-1}s_{jk}v$. The edge between $u^{-1}v$ and $u^{-1}s_{jk}v$ is labeled $t_{u^{-1}(j)} - t_{u^{-1}(k)}$ or $t_{u^{-1}(k)} - t_{u^{-1}(j)}$.

This means the polynomial $(u \cdot p)_v - (u \cdot p)_{s_{jk}v}$ is in the ideal $\langle t_j - t_k \rangle$ if and only if $p_{u^{-1}v} - p_{u^{-1}s_{jk}v}$ is in $\langle t_{u^{-1}(j)} - t_{u^{-1}(k)} \rangle$. So, the S_n -action is well-defined.

We now show this action is induced from the geometric action of S_n on G/B . We use the Borel construction of equivariant cohomology of a complex algebraic variety \mathcal{X} with the action of a torus T . If ET is the classifying bundle of T , namely a contractible space on which T acts freely, then the equivariant cohomology is defined to be $H_T^*(\mathcal{X}) = H^*(ET \times^T \mathcal{X})$, where $ET \times^T \mathcal{X}$ denotes the

quotient of the product $ET \times \mathcal{X}$ under the equivalence relation $(e, x) \sim (et, t^{-1}x)$ for all $t \in T$. For the complex torus $T = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$, one may use the classifying bundle $ET = (\mathbb{C}^\infty \setminus \{0\}) \times \cdots \times (\mathbb{C}^\infty \setminus \{0\})$. The action of $u \in S_n$ on $(e, x) \in ET \times^T G/B$ is defined by $u \cdot (e, [g]) = (eu, [u^{-1}g])$, where u acts on e by permuting its coordinates. Suppose $(et, [t^{-1}g]) \sim (e, [g])$. There is a $t' \in T$ such that $(etw, [w^{-1}t^{-1}g]) = (ewt', [(t')^{-1}w^{-1}g])$ because the group S_n normalizes the torus T . Since $(ewt', [(t')^{-1}w^{-1}g]) \sim (ew, [w^{-1}g])$, this action of S_n is well-defined on $ET \times^T G/B$.

The Borel construction of equivariant cohomology is related to the GKM construction by the inclusion $j : \mathcal{X}^T \hookrightarrow \mathcal{X}$, which for GKM spaces \mathcal{X} induces an isomorphism onto its image $j^* : H_T^*(\mathcal{X}) \hookrightarrow H_T^*(\mathcal{X}^T)$. The S_n -action on $ET \times^T G/B$ restricts to an S_n -action on $ET \times^T (G/B)^T$, which commutes with j by definition. Recall that

$$H_T^*((G/B)^T) = \oplus_{w \in S_n} H_T^*(pt) = \oplus_{w \in S_n} \mathbb{C}[t_1, \dots, t_n].$$

For each $u \in S_n$, the map $u^* : H_T^*((G/B)^T) \rightarrow H_T^*((G/B)^T)$ permutes the fixed points by u^{-1} and permutes the coordinates of the torus by u , which is precisely the action defined in this Corollary. Since $j^* \circ u^* = u^* \circ j^*$, this is the action induced by the geometric action of S_n on G/B . \square

Figure 4 gives an example of this permutation action on $H_T^*(G/B)$ when $n = 3$. An equivariant class p is denoted by a copy of the moment graph in which each

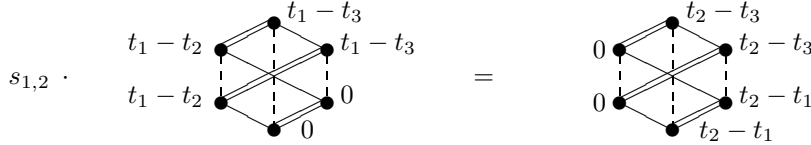


FIGURE 4. The permutation action on classes in $H_T^*(G/B)$

vertex v is labeled with the polynomial p_v . If we compute the action of $s_{2,3}$ instead of $s_{1,2}$ on the class in Figure 4, we obtain the same class we started with.

In general, the moment graph of a subvariety of the flag variety is not an induced subgraph of the moment graph for G/B . For instance, Figure 5 shows the moment



FIGURE 5. The toric variety for the decomposition into Weyl chambers in GL_3

graph for the toric variety associated to the decomposition into Weyl chambers in GL_3 . The (ordinary) cohomology of this toric variety carries a nontrivial S_3 -action that was studied in [S] and [P].

2.3. A combinatorial basis for $H_T^*(\mathcal{X})$. Once the moment graph is constructed, the GKM description of equivariant cohomology is purely combinatorial. We now discuss a combinatorial construction of a basis for the equivariant cohomology ring as a module over $H_T^*(\text{pt})$, which we call Knutson-Tao classes after [KT]. Our results are similar to those of Guillemin-Zara in [GZ1] and [GZ2] but use an underlying combinatorial model that is simpler and less restrictive.

Definition 2.11. *A combinatorial moment graph is a finite graph whose edges are labeled by linear forms and are directed so that there are no directed circuits.*

We will assume one additional condition on moment graphs in this section, which holds for moment graphs of GKM varieties for geometric reasons:

- For each $v \in \mathcal{X}^T$, if β_1, \dots, β_k label the edges directed out of v then the β_i are pairwise linearly independent.

It is a small exercise to see that that is equivalent to the following condition:

- For each $v \in \mathcal{X}^T$, if β_1, \dots, β_k label the edges directed out of v then $\langle \beta_1 \beta_2 \cdots \beta_k \rangle = \bigcap_{i=1}^k \langle \beta_i \rangle$.

If \mathcal{X} is a GKM variety, then its moment graph can be directed without circuits by choosing a suitably generic one-dimensional subtorus $T' \subseteq T$ and directing the edges according to the flow of T' (see [T1, Section 5]). Once its moment graph is directed acyclically, each edge $v \rightarrow w$ is labeled with the T -weight at v on the corresponding one-dimensional orbit. This gives (many) combinatorial moment graphs associated to \mathcal{X} .

Combinatorial moment graphs form a larger class of graphs than studied by Guillemin-Zara, including for instance moment graphs of singular GKM varieties. We note that combinatorial moment graphs need not be regular and have no particular relationship between edges directed in and out of each vertex (axiom A3 of the Guillemin-Zara axial function, [GZ1, Definition 2.1.1] or [GZ2, Definition 2.1]).

Using combinatorial moment graphs, we will show that Knutson-Tao classes are unique for a large class of varieties including Schubert varieties. For Grassmannians and flag varieties, the Knutson-Tao classes are Schubert classes. We ask whether the results of Guillemin-Zara can be extended to show that Knutson-Tao classes exist for combinatorial moment graphs.

Write $u \succeq_D u'$ if there is a directed path from u to u' in the combinatorial moment graph, possibly of length zero. For any directed graph with no directed circuits, the relation \succeq_D is a partial order on the vertices. (This partial order coincides with the Bruhat order for the moment graph of G/B in Proposition 2.8.)

We now define Knutson-Tao classes, which T. Braden and R. MacPherson also used to construct equivariant intersection cohomology [BrM] and which are homogeneous versions of the generating classes in [GZ2, Definition 2.3].

Definition 2.12. *Let \mathcal{X} be a GKM variety and let v be a T -fixed point in \mathcal{X} . A Knutson-Tao class for v is an equivariant class $(p_w^v)_{w \in \mathcal{X}^T} \in H_T^*(\mathcal{X})$ for which:*

- (1) $p_v^v = \prod_{\substack{w \in \mathcal{X}^T \text{ s.t.} \\ v \rightarrow w \text{ is an edge}}} \alpha_{vw}$, where α_{vw} is the label on the edge $v \rightarrow w$;
- (2) each nonzero p_w^v is a homogeneous polynomial with $\deg(p_w^v) = \deg(p_v^v)$; and
- (3) $p_w^v = 0$ for each w with no directed path from w to v , that is if $w \not\succeq_D v$.

Figure 6 gives three examples of Knutson-Tao classes in G/B . The reader may note that the Knutson-Tao class for the permutation (12) appeared in Figure 4.

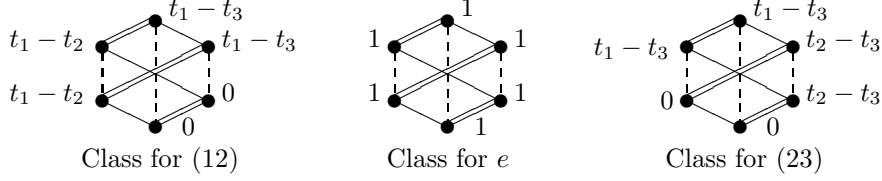


FIGURE 6. Three Knutson-Tao classes in $H_T^*(G/B)$

When they exist, Knutson-Tao classes form a basis for equivariant cohomology. The following is similar to [GZ1, Theorem 2.4.4]. The proof uses minimal elements in the moment graph; these exist because the graph is finite, directed, and acyclic.

Proposition 2.13. *Suppose that for each $v \in \mathcal{X}^T$ there exists at least one Knutson-Tao class $p^v \in H_T^*(\mathcal{X})$. Then the classes $\{p^v : v \in \mathcal{X}^T\}$ form a basis for $H_T^*(\mathcal{X})$.*

Proof. The proof will be by induction on the partial order defined by the moment graph. For each $q \in H_T^*(\mathcal{X})$ and each minimal element v in the poset, we have $q_v = c_v p_v^v$ for a unique polynomial c_v . (In fact $c_v = q_v$.) Thus $(q - c_v p^v)_v = 0$.

Let S be any subset of \mathcal{X}^T such that for each $v \in S$ and $u \prec_D v$ in the moment graph, the fixed point $u \in S$ as well. The inductive hypothesis is that for each $q \in H_T^*(\mathcal{X})$ there are unique coefficients c_v such that $(q - \sum_{v \in S} c_v p^v)_u = 0$ for each $u \in S$. For each minimal element $u' \in \mathcal{X}^T - S$, note that $(q - \sum_{v \in S} c_v p^v)_u = 0$ for all $u \prec_D u'$. The GKM conditions imply there is a unique polynomial $c_{u'}$ with

$$(q - \sum_{v \in S} c_v p^v)_{u'} = c_{u'} p_{u'}^{u'}.$$

So $(q - \sum_{v \in S} c_v p^v - c_{u'} p^{u'})_{u'} = 0$. Since $p_u^{u'} = 0$ for all $u \not\prec_D u'$, it is also true that $(q - \sum_{v \in S} c_v p^v - c_{u'} p^{u'})_u = (q - \sum_{v \in S} c_v p^v)_u = 0$ for all $u \in S$. Thus the inductive hypothesis holds for the set $S \cup \{u'\}$. There are a finite number of fixed points in \mathcal{X}^T so there is a unique expression $q = \sum_{v \in \mathcal{X}^T} c_v p^v$ for each $q \in H_T^*(\mathcal{X})$. \square

Our next results hold for a family of varieties called *Palais-Smale* varieties. Motivated by the comment in [K, page 187], we generalize the definition of Palais-Smale so that it applies to varieties that are not Hamiltonian, nor even smooth.

Definition 2.14. *The GKM variety \mathcal{X} is Palais-Smale if its moment graph can be directed so that there is an edge from v to u only if there are more edges directed out of v than out of u .*

For instance, each projective space \mathbb{CP}^n is Palais-Smale. We will see that each Schubert variety in the full flag variety is also Palais-Smale. The toric variety whose moment graph is shown in Figure 5 is *not* Palais-Smale.

Lemma 2.15. *Each Schubert variety is Palais-Smale, including G/B .*

Proof. Our convention for the moment graph for X_w is that there is a directed path from u to v only if $u > v$ in the Bruhat order. A directed circuit is a directed path from u to u . Since $u \not\prec u$, no such circuit exists in the moment graph for X_w .

There is a directed edge from $s_{jk}v$ to v only if $s_{jk}v > v$. By Proposition 2.8.1, there are $\ell(v)$ edges directed out of v in G/B and $\ell(s_{jk}v)$ edges out of $s_{jk}v$. This holds in X_w by Proposition 2.8.5. If $s_{jk}v > v$ then by definition $\ell(s_{jk}v) > \ell(v)$. \square

If \mathcal{X} is Palais-Smale, then Knutson-Tao classes are unique. The next proposition is similar to [GZ2, Theorem 2.3]. Like theirs, it could be modified to prove uniqueness of a Knutson-Tao class at one particular vertex.

Lemma 2.16. *If \mathcal{X} is Palais-Smale and $p^v = (p_w^v)_{w \in \mathcal{X}^T}$, $q^v = (q_w^v)_{w \in \mathcal{X}^T}$ are two Knutson-Tao classes corresponding to v , then $p_w^v = q_w^v$ for each $w \in \mathcal{X}^T$.*

Proof. The proof mimics [KT, Lemma 1]. Consider the class $(p^v - q^v) \in H_T^*(\mathcal{X})$. We know that $(p^v - q^v)_u = 0$ if $u = v$ or if there is no directed path from u to v . Choose a minimal vertex u_0 satisfying $(p^v - q^v)_{u_0} \neq 0$. This means both that $u_0 \succ_D v$ and that $(p^v - q^v)_{u'} = 0$ for all u' with $u_0 \succ_D u'$. In particular, the polynomial $(p^v - q^v)_{u_0}$ is in the ideal generated by the labels on the edges $u_0 \rightarrow u'$ by the GKM rules. The variety \mathcal{X} is Palais-Smale so the number of edges $u_0 \rightarrow u'$ is greater than $\deg p^v$. Since $p^v - q^v$ has degree $\deg p^v$, we conclude $(p^v - q^v)_{u_0} = 0$. \square

It is not a priori clear that *any* Knutson-Tao classes exist. We prove existence for G/B in Section 3. Existence for various families of smooth GKM varieties is proven in [GZ2].

3. PERMUTATION REPRESENTATIONS ON THE (EQUIVARIANT) COHOMOLOGY OF X_w

In this section we study permutation representations on $H_T^*(X_w)$ and $H^*(X_w)$. In Section 3.1, we show how to construct a basis of Knutson-Tao classes for $H_T^*(X_w)$ by restricting the Schubert basis for $H_T^*(G/B)$. In Section 3.2, we explicitly identify the S_n -action on $H_T^*(G/B)$ by computing how each simple transposition acts on each Schubert class. Finally, in Section 3.3, we use the formula of Section 3.2 and a restriction map $\iota : H_T^*(G/B) \rightarrow H_T^*(X_w)$ to analyze the restricted S_n -representation on $H_T^*(X_w)$ and $H^*(X_w)$.

3.1. Knutson-Tao classes in X_w and G/B .

Lemma 3.1. *Define the map $\iota : H_T^*(G/B) \rightarrow H_T^*(X_w)$ by restriction:*

$$\iota((p_v)_{v \in S_n}) = (p_v)_{[v] \in X_w}.$$

The map ι is a well-defined ring and $\mathbb{C}[t_1, \dots, t_n]$ -module homomorphism.

Proof. The moment graph of X_w is an induced subgraph of the moment graph of G/B , so for each $p \in H_T^*(G/B)$, the GKM conditions hold on $\iota(p)$. This means $\iota(p) \in H_T^*(X_w)$. By construction, ι is both a ring and module homomorphism. \square

We use the following property of equivariant cohomology [KT, Fact 2, page 10]: Let X be a T -invariant oriented cycle in \mathcal{X} , a smooth compact complex algebraic variety. Then X determines a class $[X] \in H_T^*(\mathcal{X})$ whose degree is the codimension of X in \mathcal{X} . If $u \in \mathcal{X}^T$ is not in X , then the localization of $[X]$ at u is zero.

For instance, let $[v] \in S_n$ be a permutation flag and let B^- be the group of lower-triangular invertible matrices. The closure $\overline{[B^-vB]}$ in G/B determines a class $[\Omega_v] \in H_T^*(G/B)$ called the Schubert class of v in G/B .

We give several properties of $[\Omega_v]$, which is also the class ξ^v in [KK1].

Lemma 3.2. *Write the class $[\Omega_v] \in H_T^*(G/B)$ as $[\Omega_v] = (p_u^v)_{u \in S_n}$.*

- (1) *The T -fixed points in $\overline{[B^-v]}$ are exactly those $u \in S_n$ with $u \geq v$.*
- (2) *If $v, u \in S_n$ and $u \geq v$ then the degree of p_u^v is $\ell(v)$. If $u \not\geq v$ then $p_u^v = 0$.*
- (3) *For each $v \in S_n$, the polynomial p_v^v satisfies*

$$p_v^v = \prod_{t_i - t_j \in \mathcal{I}_v} (t_i - t_j).$$

- (4) *The class $[\Omega_v]$ is the Knutson-Tao class corresponding to v in $H_T^*(G/B)$.*
- (5) *Suppose $u > v$ has $\ell(u) = \ell(v) + 1$ and that the edge from u to v is labeled $t_j - t_k$. Then the polynomial p_u^v satisfies*

$$p_u^v = \prod_{\substack{t_i - t_{i'} \in \mathcal{I}_u \\ t_i - t_{i'} \neq t_j - t_k}} (t_i - t_{i'}).$$

Proof. (1) The closure relation $\overline{[B^-u]} \subseteq \overline{[B^-v]}$ is equivalent to $u \geq v$.
(2) We use [KT, Fact 2, page 10]. The degree of $[\Omega_v]$ is the codimension of $\overline{[B^-v]}$ in G/B , namely $\ell(v)$. If $u \not\geq v$ then $u \notin \overline{[B^-v]}$ and so $p_u^v = 0$.
(3) The localization p_v^v is the product of the weights of the torus action on the normal space at $[v]$ to $[\Omega_v]$ in G/B (see [KT, Fact 3, p. 10]). The normal space at v to $\overline{[B^-v]}$ in G/B is $[U_v v]$ by, for instance, [C, Theorem G.2].
(4) The relation $u \not\geq v$ means there is no path from u to v in the moment graph of G/B . Parts 2 and 3 show that $[\Omega_v]$ is a Knutson-Tao class. Knutson-Tao classes are unique by Lemma 2.16, since G/B is Palais-Smale.
(5) Let $q = \prod_{\substack{t_i - t_{i'} \in \mathcal{I}_u \\ t_i - t_{i'} \neq t_j - t_k}} (t_i - t_{i'})$. There are $\ell(v) + 1$ edges directed out of u . One edge is to v , so there are $\ell(v)$ edges from u to vertices u' with $p_{u'}^v = 0$. The polynomial p_u^v is in the ideal generated by the labels of these $\ell(v)$ edges, so p_u^v is a scalar multiple of q . Since $u = s_{jk}v$, we have $\mathcal{I}_v \cong \{0\} \cup \mathcal{I}_{s_{jk}v} \bmod (t_j - t_k)$ by Lemma 2.5.1. By Part 3, the polynomial $q - p_v^v$ is in $\langle t_j - t_k \rangle$. Since $q \notin \langle t_j - t_k \rangle$, we conclude $p_u^v = q$. \square

The next lemma permits us to specialize calculations from G/B to X_w .

Lemma 3.3. *For each $[v] \in X_w$, restriction preserves Knutson-Tao classes:*

$$[\Omega_v]_{X_w} := \iota([\Omega_v]) \text{ is the unique Knutson-Tao class for } v \text{ in } X_w.$$

Proof. We need to check that $\iota([\Omega_v])$ satisfies the Knutson-Tao conditions.

The map ι preserves degree, so $\deg[\Omega_v]_{X_w}$ is $\ell(v)$. If $[u] \in X_w$ satisfies $u \not\geq v$ in X_w then $u \not\geq v$ in G/B , so neither the moment graph for X_w nor for G/B has a path from u to v . For each such u , the localization of $[\Omega_v]_{X_w}$ to u is zero by definition of ι . Lemma 3.2.3 proves the localization of $[\Omega_v]_{X_w}$ to v is as desired. The Knutson-Tao class is unique because X_w is Palais-Smale (Lemma 2.16). \square

When X_w is singular, the Knutson-Tao class for v need not be the geometric Schubert class corresponding to v . The precise relationship between geometric Schubert classes and Knutson-Tao classes in $H_T^*(X_w)$ is unknown.

The next corollary combines Lemma 2.8.4 with Parts 3 and 5 of Lemma 3.2. It is the heart of our later calculations.

Corollary 3.4. *Let $v, s_{i,i+1} \in S_n$ satisfy $s_{i,i+1}v > v$. Write $[\Omega_v]_{G/B} = (p_u^v)_{u \in S_n}$ and $[\Omega_{s_{i,i+1}v}]_{G/B} = (p_u^{s_{i,i+1}v})_{u \in S_n}$ for the corresponding Schubert classes. Then*

$$s_{i,i+1}p_{s_{i,i+1}v}^v = p_v^v = \frac{s_{i,i+1}p_{s_{i,i+1}v}^{s_{i,i+1}v}}{t_{i+1} - t_i}.$$

3.2. The S_n -action on $H_T^*(G/B)$. Recall that if $u \in S_n$ then u acts on $p = (p_v(t_1, \dots, t_n))_{v \in S_n} \in H_T^*(G/B)$ by

$$(u \cdot p)_v = u \cdot p_{u^{-1}v}(t_1, \dots, t_n) = p_{u^{-1}v}(t_{u(1)}, \dots, t_{u(n)}).$$

We now describe this S_n -action on $H_T^*(G/B)$ explicitly by giving a formula for the action of a simple transposition on an equivariant Schubert class. For an arbitrary permutation u , the action of u on $(p_v)_{v \in S_n}$ is obtained by factoring u into simple transpositions and then inductively applying the formula. Since the S_n -action on $H_T^*(G/B)$ is well-defined, the result is independent of the factorization of u —though that is not obvious from the formula for the action of a simple transposition!

Our methods in this section consist entirely of elementary combinatorics. For examples, the reader may wish to refer to Figures 4 and 6, which contain the calculations $s_{2,3}[\Omega_{(12)}] = [\Omega_{(12)}]$ and $s_{1,2}[\Omega_{(12)}] = [\Omega_{(12)}] + (t_2 - t_1)[\Omega_e]$.

The reader interested in the situation outside of type A_n should note that the following proof will apply immediately to general Lie type, once the generalizations of Lemma 2.5, 2.8, and 3.2 and Corollary 3.4 are given in Section 4.

Proposition 3.5. *For each Schubert class $[\Omega_w] \in H_T^*(G/B)$ and simple transposition $s_{i,i+1}$, the action of $s_{i,i+1}$ on $[\Omega_w]$ is given by*

$$s_{i,i+1}[\Omega_w] = \begin{cases} [\Omega_w] & \text{if } s_{i,i+1}w > w \text{ and} \\ [\Omega_w] + (t_{i+1} - t_i)[\Omega_{s_{i,i+1}w}] & \text{if } s_{i,i+1}w < w. \end{cases}$$

Proof. For convenience, write $s_{i,i+1}[\Omega_w] = (q_v)_{v \in S_n}$. Schubert classes form a basis for $H_T^*(G/B)$ over $\mathbb{C}[t_1, \dots, t_n]$. We expand $s_{i,i+1}[\Omega_w] = \sum c_v^w[\Omega_v]$ in terms of this basis and identify the coefficients c_v^w . To begin, the degree of $[\Omega_v]$ is $\ell(v)$ and the degree of $s_{i,i+1}[\Omega_w]$ is $\ell(w)$ by Lemma 3.2.2. This means c_v^w is zero if $\ell(v) > \ell(w)$.

Case 1: $s_{i,i+1}w > w$. We show that if $\ell(v) \leq \ell(w)$ then either $q_v = 0$ or $v = w$. Suppose v satisfies $q_v \neq 0$ and $\ell(v) \leq \ell(w)$. Since $q_v \neq 0$ we know $s_{i,i+1}v \geq w$. Recall that $\ell(s_{i,i+1}v) = \ell(v) \pm 1$. If $\ell(s_{i,i+1}v) = \ell(w)$ then in fact $s_{i,i+1}v = w$, which contradicts $s_{i,i+1}w > w$. Consequently $\ell(s_{i,i+1}v) = \ell(w) + 1 = \ell(v) + 1$. Lemma 3.2.2 implies that $s_{i,i+1}v \rightarrow w$ is an edge in the moment graph, say labeled $t_j - t_k$. If $s_{jk} \neq s_{i,i+1}$ then $s_{i,i+1}^2v > s_{i,i+1}v$ by Lemma 2.5.2. This contradicts $s_{i,i+1}v > v$. We conclude that $s_{jk} = s_{i,i+1}$ and so $v = w$. By definition $q_w = s_{i,i+1}p_{s_{i,i+1}w}^w$. By Corollary 3.4, we know $s_{i,i+1}p_{s_{i,i+1}w}^w = p_w^w$. This means $c_w^w = 1$ and $c_v^w = 0$ for all other v , namely $s_{i,i+1}[\Omega_w] = [\Omega_w]$.

Case 2: $s_{i,i+1}w < w$. First we show that $[\Omega_w] + (t_{i+1} - t_i)[\Omega_{s_{i,i+1}w}]$ agrees with $s_{i,i+1}[\Omega_w]$ in entries w and $s_{i,i+1}w$. We know that $q_w = s_{i,i+1}p_{s_{i,i+1}w}^w$ and $q_{s_{i,i+1}w} = s_{i,i+1}p_w^w$. The Knutson-Tao construction shows that $p_{s_{i,i+1}w}^w = 0$. Corollary 3.4 says $s_{i,i+1}p_w^w = (t_{i+1} - t_i)p_{s_{i,i+1}w}^{s_{i,i+1}w}$ and $p_w^w = -(t_{i+1} - t_i)p_w^{s_{i,i+1}w}$. It follows that $p_w^w + (t_{i+1} - t_i)p_{s_{i,i+1}w}^{s_{i,i+1}w} = q_u$ when $u = w$ or $u = s_{i,i+1}w$.

Suppose $v \neq s_{i,i+1}w$ is a fixed point with $q_v \neq 0$ and $\ell(v) \leq \ell(w)$. If $q_v \neq 0$ then $s_{i,i+1}v > w$. The length restrictions $\ell(v) \leq \ell(w)$ and $\ell(s_{i,i+1}v) > \ell(w)$ together with $\ell(s_{i,i+1}v) = \ell(v) \pm 1$ imply that $\ell(v) = \ell(w)$. This means the four fixed points $v, w, s_{i,i+1}v, s_{i,i+1}w$ form the fragment of the moment graph shown

in Figure 7. (The edge between v and $s_{i,i+1}w$ exists and is labeled $t_j - t_k$ since $v = s_{i,i+1}s_{j',k'}s_{i,i+1}w = s_{jk}w$ by virtue of the other three edges.)

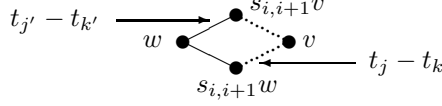


FIGURE 7. Fragment of moment graph (angles may not be to scale)

We have shown $q_v = p_v^w + (t_{i+1} - t_i)p_v^{s_{i,i+1}w}$ when $v = w$ or $v = s_{i,i+1}w$ and now show it for each $v \neq w$ such that $\ell(v) = \ell(w)$ and $v > s_{i,i+1}w$. By construction, we have $s_{j',k'} = s_{i,i+1}s_{jk}s_{i,i+1}$ and $s_{i,i+1}(t_{j'} - t_{k'}) = t_j - t_k$. Lemma 3.2.5 identifies $p_{s_{i,i+1}v}^w$ (respectively $p_v^{s_{i,i+1}w}$) as the product of the monomials in $\mathcal{I}_{s_{i,i+1}v}$ except $t_{j'} - t_{k'}$ (respectively \mathcal{I}_v and $t_j - t_k$). Note that $\mathcal{I}_{s_{i,i+1}v}$ is exactly $\{t_i - t_{i+1}\} \cup s_{i,i+1}\mathcal{I}_v$ by Lemma 2.8.4. We conclude $q_v = s_{i,i+1}p_{s_{i,i+1}v}^w = (t_{i+1} - t_i)p_v^{s_{i,i+1}w}$. Since $p_v^w = 0$ unless $s_{jk} = s_{i,i+1}$, we have $q_v = p_v^w + (t_{i+1} - t_i)p_v^{s_{i,i+1}w}$. \square

3.3. The S_n -action on $H_T^*(X_w)$ and $H^*(X_w)$. The group S_n acts on $H_T^*(X_w)$ and on $H^*(X_w)$ by restricting the S_n -action on $H_T^*(G/B)$. This section contains the main results of this paper: the representation $H_T^*(X_w)$ is not trivial over \mathbb{C} but is trivial over $\mathbb{C}[t_1, \dots, t_n]$, and the representation $H^*(X_w)$ is trivial. (These results hold for G/B since $G/B = X_{w_0}$ when w_0 is the longest permutation.)

Lemma 3.6. *For each $s \in S_n$, the action of s on the Schubert class $[\Omega_w] \in H_T^*(G/B)$ is given by*

$$s \cdot [\Omega_w] = [\Omega_w] + \sum_{v < w} c_v [\Omega_v]$$

for some polynomials $c_v \in \mathbb{C}[t_1, \dots, t_n]$ with $\deg(c_v) = \ell(w) - \ell(v)$.

Proof. If $s = s_{i,i+1}$ then the claim holds by inspection of the formula in Proposition 3.5. The permutation s can be factored as a product of simple transpositions $s_{i,i+1}$. Since $s_{i,i+1}$ preserves degree, induction on the length of s completes the proof. \square

Proposition 3.7. *The group S_n acts on $H_T^*(X_w)$ by the rule that for each $s_{i,i+1} \in S_n$ and Knutson-Tao class $[\Omega_v]_{X_w} \in H_T^*(X_w)$, we have*

$$s_{i,i+1}[\Omega_v]_{X_w} = \begin{cases} [\Omega_v]_{X_w} & \text{if } s_{i,i+1}v > v \text{ and} \\ [\Omega_v]_{X_w} + (t_{i+1} - t_i)[\Omega_{s_{i,i+1}v}]_{X_w} & \text{if } s_{i,i+1}v < v. \end{cases}$$

Proof. The restriction map $\iota : H_T^*(G/B) \rightarrow H_T^*(X_w)$ given by $\iota((p_u)_{u \in S_n}) = (p_u)_{[u] \in [S_n] \cap X_w}$ is a $\mathbb{C}[t_1, \dots, t_n]$ -module homomorphism. For each $s \in S_n$, we know that $s[\Omega_v] = [\Omega_v] + \sum_{u < v} c_u [\Omega_u]$ by Lemma 3.6. We obtain a well-defined S_n -action on $H_T^*(X_w)$ by the rule that $s \cdot [\Omega_v]_{X_w} = \iota(s[\Omega_v])$. This action restricts the S_n -representation on $H_T^*(G/B)$ in Proposition 3.5. \square

Note that this representation is not trivial as a $\mathbb{C}[S_n]$ -module. For instance, the submodule $\mathbb{C}[t_1, \dots, t_n][\Omega_e]_{X_w}$ of $H_T^*(X_w)$ is invariant under S_n . In fact, it is isomorphic to the S_n -algebra induced by the standard S_n -action on $\mathbb{C}t_1 \oplus \dots \oplus \mathbb{C}t_n$, which is not trivial. The next theorem shows this is essentially the only way in which these representations are not trivial. Write 1^d for the trivial representation in degree d .

Theorem 3.8. *The S_n -representation $H_T^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [S_n] \cap X_w} 1^{\ell(v)}$ as a graded twisted $\mathbb{C}[t_1, \dots, t_n]$ -module.*

Proof. The S_n -action preserves the degree of each class $[\Omega_v]_{X_w} \in H_T^*(X_w)$ as well as the degree of each polynomial $p \in \mathbb{C}[t_1, \dots, t_n]$. If $s \in S_n$ then we can write $s[\Omega_v]_{X_w} = [\Omega_v]_{X_w} + \sum_{u < v} c_u [\Omega_u]_{X_w}$ for some $c_u \in \mathbb{C}[t_1, \dots, t_n]$ by Lemma 3.6. Let

$$[Y_v]_{X_w} = \frac{\sum_{s \in S_n} s[\Omega_v]_{X_w}}{|S_n|}$$

so in particular $[Y_v]_{X_w}$ is S_n -invariant and has degree $\ell(v)$.

We show that the $[Y_v]_{X_w}$ generate $H_T^*(X_w)$. There are polynomials q_u with $[Y_v]_{X_w} = [\Omega_v]_{X_w} + \sum_{u < v} q_u [\Omega_u]_{X_w}$ by construction. It follows that $[Y_e]_{X_w} = [\Omega_e]_{X_w}$. This proves the inductive hypothesis that if v has length at most k then the Schubert class $[X_v]_{X_w}$ can be written in terms of the $[Y_u]_{X_w}$ for $u \leq v$. If v has length $k+1$ then $[\Omega_v]_{X_w} = [Y_v]_{X_w} - \sum_{u < v} q_u [\Omega_u]_{X_w}$ which can by induction be written in terms of the $[Y_u]_{X_w}$ for $u \leq v$. It follows that the Schubert classes, a known basis for $H_T^*(X_w)$, are generated by the $[Y_v]_{X_w}$. So $H_T^*(X_w)$ decomposes as $|X_w \cap [S_n]|$ distinct S_n -modules, each eigenspaces over $\mathbb{C}[t_1, \dots, t_n][S_n]$. \square

Corollary 3.9. *Let $\mathbb{C}[t_1, \dots, t_n]$ denote the S_n -algebra induced by the standard S_n -representation on $\mathbb{C}t_1 \oplus \dots \oplus \mathbb{C}t_n$. Then $H_T^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [S_n] \cap X_w} 1^{\ell(v)} \otimes \mathbb{C}[t_1, \dots, t_n]$ as a graded $\mathbb{C}[S_n]$ -module.*

The S_n -action on $H_T^*(X_w)$ gives rise to a S_n -action on $H^*(X_w)$.

Theorem 3.10. *Consider $H^*(X_w)$ as a $\mathbb{C}[S_n]$ -module with the S_n -action inherited from $H_T^*(X_w)$. Then $H^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [S_n] \cap X_w} 1^{\ell(v)}$ as a graded $\mathbb{C}[S_n]$ -module.*

Proof. The ring $H^*(X_w)$ is isomorphic to $H_T^*(X_w)/\langle t_1, \dots, t_n \rangle$ (see [GKM, Equation 1.2.4]). Consequently, the S_n -action on $H_T^*(X_w)$ induces an S_n -action on $H^*(X_w)$ given by $s_{i,i+1}[\Omega_v]_{X_w} = [\Omega_v]_{X_w}$ for each $s_{i,i+1} \in S_n$ and $[v] \in X_w \cap [S_n]$. This representation is trivial. \square

3.4. Divided difference operators. This action of S_n on $H_T^*(X_w)$ gives rise to divided difference operators on the equivariant cohomology ring.

Definition 3.11. *For each $i = 1, \dots, n-1$, the i^{th} (left) divided difference operator D_i is defined by*

$$D_i(p) = \frac{p - s_{i,i+1} \cdot p}{t_i - t_{i+1}}$$

for each $p \in H_T^*(X_w)$.

Note that the left divided difference operator uses the S_n -action on equivariant classes rather than the S_n -action on the polynomials obtained by localizing equivariant classes.

We prove that the divided difference operators are well-defined by using the following stronger result. Let $\partial_i : \mathbb{C}[t_1, \dots, t_n] \rightarrow \mathbb{C}[t_1, \dots, t_n]$ denote the (ordinary) divided difference operator defined by $\partial_i(p) = \frac{p - s_i p}{t_i - t_{i+1}}$.

Proposition 3.12. *Let $p \in H_T^*(X_w)$ be an equivariant class whose expansion in terms of the basis of Knutson-Tao classes is $p = \sum c_v[\Omega_v]$. Then*

$$D_i(p) = \sum \partial_i(c_v)[\Omega_v] + \sum_{v: s_i v < v} s_i(c_v)[\Omega_{s_i v}].$$

Proof. The S_n -action on $H_T^*(X_w)$ is \mathbb{C} -linear, so $D_i(p) = \sum D_i(c_v[\Omega_v])$. Proposition 3.7 shows that $D_i(c_v[\Omega_v])$ is $\partial_i(c_v)[\Omega_v]$ if $s_i v > v$ and is $\partial_i(c_v)[\Omega_v] + s_i(c_v)[\Omega_{s_i v}]$ if $s_i v < v$. \square

In particular $D_i(p)$ is an element of $H_T^*(X_w)$ whenever p is in $H_T^*(X_w)$.

Every Schubert class of G/B can be obtained by performing a sequence of divided difference operators on the class of the longest permutation $[X_{w_0}]$. This is not true for general Schubert varieties. For instance, no sequence of divided difference operators performed on the highest class $[X_{s_1 s_2}]$ of the Schubert variety $X_{s_1 s_2}$ gives the class $[X_{s_2}]$.

The divided difference operator D_i is not the same as the operator defined by Bernstein-Gelfand-Gelfand and Demazure [BGG], [D], though the formulas are similar. Another divided difference operator was defined by Kostant-Kumar in [KK1]: for each $p \in H_T^*(G/B)$ and each $v \in S_n$ let

$$(\partial_i p)(v) = \frac{p(v) - p(vs_i)}{-v(t_i - t_{i+1})}.$$

Arabia proved in [A] that the operator ∂_i is the same morphism as the Bernstein-Gelfand-Gelfand/Demazure divided difference operator. The formulas differ because Bernstein-Gelfand-Gelfand/Demazure use a different presentation of the ring $H_T^*(G/B)$ than Kostant-Kumar (and we) do.

4. GENERAL LIE TYPE

We now describe these results in arbitrary Lie type. Our exposition is brief: we assume our reader is familiar with the general theory and only indicate the proofs whose generalization is not immediate.

In this section, G is a complex reductive linear algebraic group, B a Borel subgroup, and T a maximal torus contained in B . The full flag variety is G/B . We write an element of the flag variety as $[g]$. The Weyl group is $W = N(T)/T$. The set of simple reflections in W is written S and the set of all reflections is $R = \bigcup_{w \in W} wSw^{-1}$. The roots are denoted Φ , with positive roots Φ^+ and negative roots Φ^- . The elements in R are the reflections associated to the positive roots. We write $s_\alpha \in R$ to denote the reflection associated to $\alpha \in \Phi^+$.

There is a partial order on W called the Bruhat order and defined by the condition that $w \geq v$ if and only if $\overline{Bw} \supseteq Bv$. This is equivalent to the condition that there is a reduced factorization $w = s_{i_1} \cdots s_{i_k}$ with each $s_{i_j} \in S$ such that v is the (ordered) product of a substring of the s_{i_j} . The length of w is the minimal k required to factor $w = s_{i_1} \cdots s_{i_k}$ as a product of simple transpositions $s_{i_j} \in S$. The length of w is denoted $\ell(w) = k$.

The next few results are combinatorial properties of W . This next proposition complements [BGG, Lemma 2.4].

Proposition 4.1. *Suppose $s_\alpha \in R$ and $w \in W$ satisfy $\ell(s_\alpha w) = \ell(w) + 1$. Then*

- (1) $s_\alpha w > w$; and

(2) if s_i is a simple reflection with $s_i \neq s_\alpha$ and $s_i w > w$, then $s_i s_\alpha w > s_\alpha w$.

Proof. The first part follows because either $s_\alpha w > w$ or $s_\alpha w < w$, and the Bruhat order respects length. For the second part, suppose $s_i s_\alpha w < s_\alpha w$. By the exchange property [Bou, IV.1.5] or [BB, Corollary 1.4.4], there is a reduced expression for $s_\alpha w$ that begins with s_i , say $s_i s_{i_1} \cdots s_{i_k}$. Since $s_\alpha w > w$ and $\ell(s_\alpha w) = \ell(w) + 1$, we can obtain a reduced factorization of w by erasing one of the simple transpositions in the string $s_i s_{i_1} \cdots s_{i_k}$. If it were not s_i then $s_i w < w$. So $s_i s_\alpha w > s_\alpha w$ unless $s_\alpha = s_i$. \square

The following set is well-known in the literature (e.g. [Bou, VI.1.6, Proposition 17]) but seems not to have a concise name.

Definition 4.2. Given $w \in W$, the inversions corresponding to w are the roots

$$\mathcal{I}_w = \{\alpha \in \Phi^+ : w^{-1}\alpha \in \Phi^-\} = \Phi^+ \cap w\Phi^-.$$

For instance, each simple transposition s_i has a unique inversion $\mathcal{I}_{s_i} = \{\alpha_i\}$. There are $\ell(w)$ elements in the set \mathcal{I}_w .

Proposition 4.3. If $s_i w > w$ then $\mathcal{I}_{s_i w} = s_i \mathcal{I}_w \cup \{\alpha_i\}$.

Proof. The definition of \mathcal{I}_w implies that $(w^{-1}s_i)s_i \mathcal{I}_w \subseteq \Phi^-$. By the exchange property [Bou, IV.1.5] or [BB, Corollary 1.4.4], there is a reduced factorization for $s_i w$ beginning with s_i . It is a small exercise (proven in, e.g., [H, Corollary 10.2.C]) to see that this implies that $(s_i w)^{-1}(\alpha_i)$ is negative. We conclude that $\alpha_i \in \mathcal{I}_{s_i w}$ and that $\alpha_i \notin \mathcal{I}_w$. The simple reflection s_i negates α_i and permutes the other positive roots, so $s_i \mathcal{I}_w \subseteq \Phi^+$. We have shown that the set $s_i \mathcal{I}_w \cup \{\alpha_i\}$ consists of distinct roots and is contained in $\mathcal{I}_{s_i w}$. Since both sets contain $\ell(w) + 1$ roots, they are in fact equal. \square

Proposition 4.4. If $s_\alpha w > w$ and $\ell(s_\alpha w) = \ell(w) + 1$ then

$$\mathcal{I}_{s_\alpha w} = \{\alpha\} \cup (\Phi^+ \cap s_\alpha \mathcal{I}_w) \cup (\mathcal{I}_w \cap s_\alpha \Phi^-).$$

Proof. The root α must be in exactly one of \mathcal{I}_w and $\mathcal{I}_{s_\alpha w}$. Since $s_\alpha w > w$, we have $\alpha \in \mathcal{I}_{s_\alpha w}$ (see [C, Theorem F4] or Proposition 4.6.2.). Note that if $\beta \in \mathcal{I}_w$ then $(s_\alpha w)^{-1}s_\alpha(\beta)$ is a negative root. If $s_\alpha(\beta) \in \Phi^+$ then $s_\alpha(\beta) \in \mathcal{I}_{s_\alpha w}$. Suppose instead that $s_\alpha(\beta) \in \Phi^-$. There is a positive integer c_β such that $s_\alpha(\beta) = \beta - c_\beta \alpha$ by definition of reflections. On the one hand, we know that $\beta = s_\alpha(\beta) + c_\beta \alpha$ is a positive root. On the other hand, we know that $w^{-1}s_\alpha(s_\alpha(\beta) + c_\beta \alpha) = w^{-1}(\beta) + c_\beta w^{-1}(-\alpha)$. We conclude that β is in $\mathcal{I}_{s_\alpha w}$, since $w^{-1}(\beta)$ and $w^{-1}(-\alpha)$ are both negative and c_β is positive.

We have shown that the map sending $\beta \in \mathcal{I}_w$ to $\begin{cases} s_\alpha(\beta) & \text{if } s_\alpha(\beta) \in \Phi^+ \\ \beta & \text{else} \end{cases}$

has image in $\mathcal{I}_{s_\alpha w}$. It is an injection because it is invertible. By comparing cardinalities, we know that $\mathcal{I}_{s_\alpha w}$ consists of the image of this map together with exactly one other root. This proves the claim. \square

Corollary 4.5. If $s_\alpha w > w$ and $\ell(s_\alpha w) = \ell(w) + 1$ then

$$\mathcal{I}_{s_\alpha w} = \{\alpha\} \cup s_\alpha \mathcal{I}_w \pmod{(\alpha)} \text{ (with multiplicity).}$$

Proof. If β in \mathcal{I}_w and $s_\alpha(\beta) \notin \mathcal{I}_{s_\alpha w}$ then by the previous proposition, there is a positive integer c_β with $\beta = s_\alpha(\beta) + c_\beta \alpha \in \mathcal{I}_{s_\alpha w}$. In this case $\beta \equiv s_\alpha(\beta) \pmod{\alpha}$. \square

J. Carrell described the moment graph for Schubert varieties in G/B in [C, Theorem F]. We give his result here for the convenience of the reader. The root subgroup corresponding to α is written U_α .

Proposition 4.6. (1) *The vertices for the moment graph for G/B are exactly the flags $[w]$ for $w \in W$.*
 (2) *There is an edge between $[v]$ and $[w]$ in the moment graph if and only if $v^{-1}w \in R$. If $v^{-1}w = s_\alpha$ then the edge is labeled α and corresponds to the one-dimensional T -orbit $[U_\alpha w] \cup [U_\alpha v]$. The edge between $[s_\alpha v]$ and $[v]$ is directed from the element of greater length to the element of lesser length.*
 (3) *The edges directed out of $[w]$ are labeled exactly by the roots in \mathcal{I}_w .*
 (4) *The moment graph for the Schubert variety $X_w = \overline{[Bw]}$ is the subgraph induced by the elements $[v] \in [W] \cap X_w$ from the moment graph for G/B .*

Write $H_T^*(\text{pt})$ as the polynomial ring in the simple roots $\mathbb{C}[\alpha_1, \dots, \alpha_n]$. The Weyl group acts on $H_T^*(\text{pt})$ by extending the coadjoint W -action on Φ , namely $w \in W$ acts on $p(\alpha_1, \dots, \alpha_n) \in H_T^*(\text{pt})$ by $w \cdot p(\alpha_1, \dots, \alpha_n) = p(w(\alpha_1), \dots, w(\alpha_n))$. The equivariant cohomology of each Schubert variety $H_T^*(X_w)$ is a module over $H_T^*(\text{pt})$.

In particular, let $[\Omega_v]$ be the class in $H_T^*(G/B)$ determined by the closure $\overline{[B^{-1}vB]}$ in G/B . The previous results imply that $[\Omega_v]$ is the Knutson-Tao class for v .

Proposition 4.7. *Let $v \in W$ and write $[\Omega_v] \in H_T^*(G/B)$ as $[\Omega_v] = (p_u^v)_{u \in W}$.*

- (1) *If $u \not\geq v$ then $p_u^v = 0$. If $u \geq v$ then p_u^v has degree $\ell(v)$.*
- (2) *The localization of $[\Omega_v]$ to v is $p_v^v = \prod_{\alpha \in \mathcal{I}_v} \alpha$.*
- (3) *The class $[\Omega_v]$ is the unique Knutson-Tao class for v in $H_T^*(G/B)$.*
- (4) *If $u > v$ satisfies $u = s_\alpha v$ and $\ell(u) = \ell(v) + 1$ then*

$$p_u^v = \frac{1}{\alpha} \prod_{\beta \in \mathcal{I}_u} \beta.$$

- (5) *If $s_i \in S$ satisfies $s_i v > v$ then*

$$s_i p_{s_i v}^v = p_v^v = \frac{s_i p_{s_i v}^{s_i v}}{-\alpha_i}.$$

Proof. As in type A_n , the closure $\overline{[B^{-1}vB]}$ contains exactly the T -fixed points $[w]$ with $w \geq v$. The normal space to $\overline{[B^{-1}vB]}$ at v is given by $[Bv]$, whose torus weights are indexed by \mathcal{I}_v . So, the proofs of Parts 1, 2, and 3 are identical to those in Lemma 3.2. The proof of the other parts is closely parallel to those for type A_n . Write $q = \frac{1}{\alpha} \prod_{\beta \in \mathcal{I}_u} \beta$. The polynomial $p_u^v \in \langle q \rangle$ by the GKM conditions, so $p_u^v = cq$ for some (complex) constant c . Corollary 4.5 shows $q - p_v^v \in \langle \alpha \rangle$, so $c = 1$. This proves Part 4. Part 5 follows directly from Part 2 and Proposition 4.3. \square

We now describe a group action of W on $H_T^*(G/B)$ induced by the action of $u \in W$ on $[v] \in [W]$ given by $u \cdot [v] = [u^{-1}v]$. Unlike the case when $G = GL_n(\mathbb{C})$, this group action is not well-defined on the entire flag variety, since $u \in W = N(T)/T$ is defined only up to T . However, the action does induce a graph automorphism of the moment graph of G/B . This action arises from the action of $u \in W$ on $ET \times^T G/B$ given by $u \cdot (e, [g]) = (eu, [u^{-1}g])$ just as in Corollary 2.10.

Proposition 4.8. *The group W acts on $H_T^*(G/B)$ according to the rule that if $u \in W$ and $p = (p_v)_{v \in W} \in H_T^*(G/B)$ then for each $v \in W$*

$$(u \cdot p)_{v \in W} = u \cdot p_{u^{-1}v}(\alpha_1, \dots, \alpha_n) = p_{u^{-1}v}(u(\alpha_1), \dots, u(\alpha_n)).$$

Proof. We give the main step of the proof in detail; the rest is exactly as in Corollary 2.10. The action of $u^{-1} \in W$ on G/B given by $u^{-1} \cdot [g] = [u^{-1}g]$ is defined on each one-orbit $[U_\alpha v]$ since each T -one-orbit is T -closed. The action satisfies $u^{-1} \cdot [U_\alpha v] = [U_{u^{-1}(\alpha)} u^{-1}v]$. If $u^{-1}(\alpha) \in \Phi^+$ then this is a one-orbit between $[u^{-1}v]$ and $[s_{u^{-1}(\alpha)} u^{-1}v]$ in G/B . If $u^{-1}(\alpha) \in \Phi^-$ then $[U_{-u^{-1}(\alpha)} s_{u^{-1}(\alpha)} u^{-1}v]$ is a one-orbit between $[s_{u^{-1}(\alpha)} u^{-1}v]$ and $[u^{-1}v]$ in G/B . In either case, there is an edge between v and $s_\alpha v$ if and only if there is an edge between $u^{-1}v$ and $u^{-1}s_\alpha v = s_{u^{-1}(\alpha)} u^{-1}v$. \square

The propositions included in this section are exactly those needed in Proposition 3.5. The proof of Proposition 3.5 applies in our more general setting once the appropriate notational changes are made: α_i for $t_i - t_{i+1}$, s_i for $s_{i,i+1}$, α for $t_j - t_k$, and s_α for s_{jk} . We give the statement of the general theorem here.

Theorem 4.9. *For each Schubert class $[\Omega_w] \in H_T^*(G/B)$ and simple transposition s_i , the action of s_i on $[\Omega_w]$ is given by*

$$s_i[\Omega_w] = \begin{cases} [\Omega_w] & \text{if } s_i w > w \text{ and} \\ [\Omega_w] - \alpha_i[\Omega_{s_i w}] & \text{if } s_i w < w. \end{cases}$$

The restriction of this action to $H_T^*(X_w)$ and the analysis of the W -action on $H_T^*(X_w)$ and $H^*(X_w)$ is also identical to the case of type A_n . The statement is:

Theorem 4.10. *Let X_w be a Schubert variety in G/B . The trivial representation of W in degree d is denoted 1^d , and the W -algebra induced by the coadjoint action on Φ is denoted $\mathbb{C}[\alpha_1, \dots, \alpha_n]$.*

- (1) $H_T^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [W] \cap X_w} 1^{\ell(v)} \otimes \mathbb{C}[\alpha_1, \dots, \alpha_n]$ as a graded twisted $\mathbb{C}[W]$ -module;
- (2) $H_T^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [W] \cap X_w} 1^{\ell(v)}$ as a graded twisted module over $\mathbb{C}[\alpha_1, \dots, \alpha_n][W]$;
- (3) and $H^*(X_w)$ is isomorphic to $\bigoplus_{[v] \in [W] \cap X_w} 1^{\ell(v)}$ as a graded $\mathbb{C}[W]$ -module.

The left W -action also gives rise to (left) divided difference operators on the equivariant cohomology of Schubert varieties in general type.

Definition 4.11. *For each $s_i \in W$, the divided difference operator D_i is defined by*

$$D_i(p) = \frac{p - s_i \cdot p}{\alpha_i}$$

for each $p \in H_T^*(X_w)$.

As before, there is an explicit formula for the action of D_i . Let $\partial_i : \mathbb{C}[\alpha_1, \dots, \alpha_n] \rightarrow \mathbb{C}[\alpha_1, \dots, \alpha_n]$ denote the i^{th} (ordinary) divided difference operator.

Proposition 4.12. *If $p = \sum c_v[\Omega_v]$ is an equivariant class in $H_T^*(X_w)$ then*

$$D_i(p) = \sum \partial_i(c_v)[\Omega_v] + \sum_{v: s_i v < v} s_i(c_v)[\Omega_{s_i v}].$$

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